

The Drude Model

Miguel Kiwi^{*1}, Jaime Rössler¹

¹Universidad de Chile, Facultad de Ciencias, 780-0024, Santiago, Chile.

Received on June 03, 2024. Revised on July 29, 2024. Accepted on July 31, 2024.

In 1900, Paul Drude made a groundbreaking advancement with a model to describe the electrical conductivity of metals. Over 120 years later, his model remains widely used and featured in most introductory textbooks. Drude developed this model shortly after the discovery of the electron, during a time when the precise atomic structure was still debated, the Pauli exclusion principle was unknown, and well before Fermi-Dirac statistics. In this discussion, we focus on the assumptions underlying Drude's model, mainly the choice of the electron mean collision time, analyze the physics it employs, and evaluate how well it fits experimental results of Ohm's law, both for static and time-dependent fields. We also explore the insights it provides into thermal effects and briefly address how quantum effects can be incorporated. Overall, we highlight why Drude's achievement is a remarkable success story that merits thorough examination and admiration.

Keywords: electrical conductivity, statistical mechanics.

1. Introduction

Drude's [1] work in 1900 set a major milestone on the description of the conductivity of metals. That it was developed before major ingredients, like the concept of Fermi velocity, the Pauli exclusion principle, and accurate atomic sizes were well established, is worth a closer analysis. In fact, Drude was one of the pioneers in the transition from "classical" to "modern" physics; that is, in the evolution from a mere description of the properties of matter (in our case metals) to a formal understanding on the basis of microscopic atomic scale components. Moreover, he not only addressed metal DC conductivity, but also AC conductivity, partially explained the Wiedemann-Franz law, thermoelectric effects (Peltier and Seebeck), and the electron specific heat (with an error of the order of the thermal velocity relative to the electron Fermi velocity). The model was refined by H A Lorentz [2] in 1905, and later by many other pioneers, including such figures as Sommerfeld, Bloch, and Bethe. Actually, the first page of Ashcroft and Mermin's text [3] states that the "remaining conceptual puzzles defined the problems" that the theory of metals was to solve during the next quarter century.

His work and that of Lorentz [2, 4] are closely intertwined. In fact the Lorentz extended the understanding of electronic transport phenomena, and related his name to the Wiedeman-Franz law, derived empirically in 1853, through the relation

$$\frac{\kappa}{\sigma T} = L_0, \quad (1)$$

where L_0 is the Lorentz number, κ and σ are the thermal and electric conductivities, and T the temperature.

The success of Drude relies heavily on the adoption of a "mean collision time" τ , instead of the seemingly more logical choice of the "mean free path" λ . The introduction of τ leads directly to Ohm's law, while the use of λ demands a rather careful analysis. However, the use of τ requires electron speeds of the order of 100 km/s to describe experiments, even at zero temperature. These speeds, in turn, only appear naturally after the Pauli principle was well established, and the consequent Fermi velocities, are introduced in the picture; concepts unknown in the early twentieth century,

Avogadro postulated in 1811 that equal volumes of gases at the same temperature and pressure contain the same number of molecules. The Avogadro number only was precisely determined almost a century later, in 1908 by Perrin [5, 6], but it is fundamental to determine inter-atomic distances in metals. Also, other key ingredients were not available to Drude at the time he developed his model.

The basic hypothesis he made were: a) that metal valence electrons form a non-interacting gas. This constitutes a major step forward, since at that time J.J. Thomson's plum pudding atomic model [7] assumed electrons at rest; b) electron-electron interactions are neglected. This hypothesis might have been criticized at that time, but was vindicated once the quantum fermionic character of the electrons became established; c) electrons are assumed to collide with hard core ions, and follow free particle like straight trajectories between collisions; d) the effect of every collision is to thermalize electron velocities, governed by a Boltzmann distribution, with the energy loss accounting for resistive metal

*Correspondence email address: m.kiwi.t@gmail.com

heating (Joule heating); and e) Drude characterized the dissipation process by a relaxation time τ , of the order of magnitude of the average time between two electron-ion collisions. Although according to assumptions a)–d) it seemed that the mean free path was the logical choice, but he preferred the relaxation time τ . In some way the choice of τ acts as a black box that “hides” the lack of knowledge of the quantum behavior of the microscopic world, prevalent at that time. This assumption also implies the damped Newton equation of motion

$$\frac{d}{dt}\vec{p}(t) = \vec{f}(t) - \frac{1}{\tau}\vec{p}(t), \quad (2)$$

where $\vec{p}$ is the electron momentum, and $\vec{f}$ the force exerted on it.

2. Analysis

From the asymptotic solution of equation 2 Ohm's law

$$\vec{J} = \sigma \vec{E}, \quad (3)$$

can be readily derived. Here $\vec{E}$ is the static electric field, σ the conductivity and $\vec{J}$ the current density. In turn

$$\sigma = \frac{e^2 \tau}{m} n_e, \quad (4)$$

where e is the carrier charge (electrons), n_e the electron density, m their mass, n_e the charge density, and τ the “mean collision time”; as already mentioned, for the collisions the crystal lattice atoms are assumed to be hard spheres at rest. To obtain a linear relation between σ and $\vec{E}$ (equation 3) it is necessary that

$$e\mathcal{E}\lambda \ll \frac{3}{2}k_B T, \quad (5)$$

which implies that the potential drop along a mean free path has to be small ($\Delta V \ll k_B T$). In the context of the Drude model this hypothesis is quite suitable. For example, at room temperature, and taking for λ the inter-atomic distance of 2×10^{-10} m it holds that $e\mathcal{E}\lambda/k_B T = 7.7 \times 10^{-6}$ for $\mathcal{E}=1,000$ V/m. But even this $\mathcal{E}$ -value is much too large, due to Joule heating, since for a 2 mm diameter copper wire it would dissipate 440,000 cal/cm·s.

The electron density n_e could not be estimated quite accurately at that time, since Loschmidt overestimated the Avogadro number by an order of magnitude. But since it is the “mean collision time” τ that plays the major role we will analyze this issue in detail. In accordance to Drude's hypothesis [1] of an electron gas at a temperature T the collision time τ is identified with the collision time between two scattering centers at a distance λ apart

$$\tau = \frac{\lambda}{v_e(T)} = \lambda \sqrt{\frac{m_e}{3k_B T}} \quad (6)$$

where $v_e(T)$ is the thermal velocity. If we take the today values for the above parameters one obtains $v_e(T = 14^\circ\text{C}) \approx 1.2 \times 10^5$ m/s, $\lambda(\text{Cu}) \approx 2.28 \times 10^{-10}$ m, and $\tau \approx 2 \times 10^{-15}$ s. Since an electron in a rather strong field of 10^3 V/m during the above estimated time τ only increases its speed by $\Delta v_e \approx 0.37$ m/s, these results are compatible with Drude's model. A more detailed study of equation 2 requires a characteristic electron collision time; in which case a linear response results if the condition $e\mathcal{E}\lambda \ll \frac{3}{2}k_B T$ is satisfied. Combining equations 4 and 6 we obtain

$$\rho(T) = \frac{1}{\sigma(T)} \propto \sqrt{T} \quad (7)$$

which implies a negative curvature of $\rho(T)$ which is not observed experimentally. We use Drude's model, with the values of today: $e \approx 1.60 \times 10^{-19}$ C, $m_e \approx 9.11 \times 10^{-31}$ kg, $n_e(\text{Cu}) \approx 8.44 \times 10^{28}$ m $^{-3}$, and $\lambda(\text{Cu}) \approx 2.28 \times 10^{-10}$ m, and $\tau(\text{Cu } 20^\circ\text{C}) \approx \times 10^{-15}$. Introducing these values in equation 4 we conclude that $\sigma[\text{Cu}(20^\circ\text{C})] \approx 4.64 \times 10^6$ ($\Omega \cdot \text{m}$) $^{-1}$, while the actual value is 5.9×10^7 ($\Omega \cdot \text{m}$) $^{-1}$, which is only an order of magnitude off. But, in 1900 the values of the parameters we used above were quite uncertain, and the result must have seemed surprisingly good.

3. Ohm's Law and the Drude Hypothesis

While today Drude's derivation of Ohm's law seems quite natural he had to carefully consider which hypothesis to make for the behavior of electrons, in an environment where the understanding of the atomic structure was quite dim. We recall that the plum pudding model of the atom by J.J. Thomson avoided electron motion by their insertion into an extended positive background, in order to avoid electromagnetic radiation. Only after the experiments by Rutherford in 1909, and the Bohr quantum atomic model of 1915, a slightly more realistic picture started to emerge. That Drude introduced the idea of an electron gas constitutes a major achievement in itself and deserves full recognition, since in the pre-quantum era the prevailing notion was of immobile electrons. Around 1900 it would have seemed more natural to assume electrons at rest, accelerated by an external electric field that would carry them to a neighboring atom, where it would return to rest. If we name λ the traveled inter-atomic distance then the drift velocity

$$v_{drift} = -\sqrt{\frac{\lambda e \mathcal{E}}{2m_e}} \rightarrow J = ne\sqrt{\frac{\lambda e \mathcal{E}}{2m_e}} \propto \sqrt{\mathcal{E}}. \quad (8)$$

equation 8 is an expression quite different from Ohm's law. Therefore, a more detailed understanding of the relaxation time τ hypothesis is warranted. In fact, the introduction of a dissipation process also was no easy matter; once electron-electron collisions are excluded only inelastic electron-ion collisions are plausible. But,

since crystal ordering was not yet established, the natural association was to relate λ to poorly known inter-atomic distances. Therefore, in 1900 the choice λ as the fundamental parameter must have seemed the logical choice. However, in the context of the Drude electron gas assumption, λ is related to the relaxation time τ by

$$\tau = \lambda/v(T), \quad (9)$$

and to $\rho(T) \propto \sqrt{T}$ which does not agree with experimental results. This led Drude to the choice of the relaxation time τ as the basic parameter; in τ all the unknowns related to the microscopic world were hidden. Fortunately, this “black box” was later successfully opened by quantum mechanics.

But experiments had already shown this T -dependence to be incorrect, since in 1838 Lenz [8, 9], and later other authors [10–12], carried out measurements that suggested a linear T dependence with a slight positive curvature (see Fig. 1). The expression

$$\rho(T) = \rho_0(1 + aT + bT^2) \quad (10)$$

with a and b constants was used, without much theoretical justification beyond a least square fit [8–10]. Siemens [13] adopted this expression. However, not being able to fit the data he later postulated that [14]

$$R(T) = \alpha\sqrt{T} + \beta T + \gamma, \quad (11)$$

where α, β and γ are constants. This led to a lengthy dispute with Matthiessen [15] on the establishment of an electric resistance standard, but all the available experimental evidence [10–12] showed a positive curvature of ρ vs T . Moreover, in 1870 Matthiessen correctly pointed out the relevance of impurities, and formulated his well known “Matthiessen rule” that is still used today. It is given by

$$\rho = \rho_0 + \rho(T), \quad (12)$$

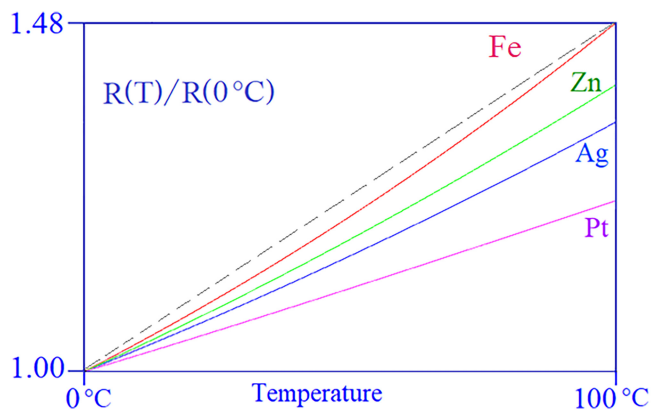


Figure 1: Resistivity vs. temperature for several metals according to measurements made by Lenz [8, 9]. The iron resistivity is compared to the dashed straight line to underscore the positive curvature of $R_{Fe}(T)$.

where ρ_0 corresponds to the residual resistivity, mainly due to impurities, and $\rho(T)$ is mostly due to electron-phonon scattering.

It is worth mentioning that in the nineteenth century cryogenics techniques were not available; therefore, no significantly temperatures lower than 0°C were accessible for experiments; however, in 1892 Wróblewski and K. Olszewski [16–18] were able to liquefy Nitrogen ($T = -195.8$ K), which allowed Dewar and Fleming [11, 12] to perform their measurements in a temperature range $-200 < T < +100^\circ\text{C}$. Their measurements show a quasi-linear ρ vs. T dependence with moderate positive curvatures. When in 1908 Kamerlingh Onnes succeeded liquefying He a whole new area of physics was born, allowing the investigation of metal properties closer to 0 K.

Nevertheless, the original Drude model has several shortcomings: i) it fails to describe the “anomalous Hall effect”, which suggests that electrons could have positive charge; ii) it does not provide an understanding of semiconductor conductivity, which grows as an exponential function of T^{-1} ; iii) it does not shed light on the description provided by Matthiessen’s rule. All these difficulties require Quantum Mechanics and electron band theory to be properly understood. However, and in spite of this fact, Drude’s approach goes even further. Using equation 2, and based on the work by Hertz on electromagnetic waves of angular frequency ω it holds that

$$\mathcal{E}(t) = \Re(\mathcal{E}_0 \exp i\omega t) \quad (13)$$

$$\vec{\mathbf{J}}(t) = \Re(\sigma(\omega)\mathcal{E}_0 \exp i\omega t), \quad (14)$$

where $\Re$ stands for real part. This leads to

$$\sigma(\omega) = \sigma_0 \left(\frac{1}{1 + \omega^2\tau^2} + i\omega\tau \frac{1}{1 + \omega^2\tau^2} \right). \quad (15)$$

He also analyzed the thermal conductivity $\vec{\mathbf{J}}_q$ of metals and their specific heat C_V , obtaining

$$C_V = \frac{3}{2}n_e k_B T \quad (16)$$

$$\vec{\mathbf{J}}_q = -\frac{1}{3}v^2\tau C_V \nabla T = -\kappa \nabla T, \quad (17)$$

where v^2 is the mean square velocity and κ the thermal conductivity. He next successfully applied these results to derive for the Wiedemann-Franz law the expression

$$\frac{\kappa}{\sigma T} = L_0 = \frac{3}{2} \left(\frac{k_B}{e} \right)^2 \approx 1.11 \times 10^{-8} \frac{\text{W}\Omega^2}{\text{K}}, \quad (18)$$

where the Lorentz number L_0 is quite close to experimental results, which are only 2 or 3 times larger. This excellent fit is in part due to the τ cancellation in deriving the Wiedemann-Franz law. In addition, and also in 1900, Drude obtained results on galvanomagnetic effects [19].

4. Thermoelectric Effects

It is quite likely that Drude, in part motivated by the work of Seebeck and Peltier on thermo-electricity half a century earlier, postulated the existence of an electron gas since these effects can be described under that hypothesis. We will do so by means of a qualitative analysis, again based on the mean free path concept. Consider a section S metal rod oriented in the x direction, subject to a linear temperature decrease from T_+ to T_- as x increases; at the rod center $T(x=0) = (T_+ + T_-)/2$. Assume now that electrons moving to the right in the interval $x, x + \lambda$ are thermalized at $T = T(x=0)$, and move at an average speed $\langle v(x) \rangle$; while those moving to the left are at $T(x-\lambda)$ and move with $\langle v(x-\lambda) \rangle$. Here

$$\langle v(T(x)) \rangle = \frac{\int_0^\infty dv v e^{-mv^2/2k_B T}}{\int_0^\infty dv e^{-mv^2/2k_B T}} = \sqrt{\frac{2k_B T}{\pi m}}. \quad (19)$$

The current is the difference between the flux moving to the right and to the left. During δt the number of electrons crossing the plane at $x + \lambda$ is $\frac{1}{2} n_e S \langle v(x + \lambda) \rangle \delta t$, while the number crossing in the opposite direction is $\frac{1}{2} n_e S \langle v(x) \rangle \delta t$, and the factor $1/2$ is due to the fact that only half of the carriers move either right or left. Consequently, the net current density is

$$J_T = -\frac{1}{2} n_e e (\langle v(T + \delta T) \rangle - \langle v(T) \rangle). \quad (20)$$

The - sign above is due to the negative electron charge, an information not available to Drude then. Since we assumed a linear temperature gradient

$$\delta T = -\frac{\partial T}{\partial x} \lambda \quad (21)$$

one obtains, combining the above relations

$$\frac{\partial v(T)}{\partial T} = \frac{v(T)}{2T} \quad (22)$$

$$\vec{J}_T = \frac{1}{4} n_e e \vec{v}(T) \frac{\partial T}{\partial x} \lambda \quad (23)$$

We now write this result in in terms of the relaxation time $\tau = \lambda/v_T$, which reads

$$\vec{J}_T = \kappa \left(\frac{\partial T}{\partial x} \right), \quad (24)$$

where the thermal conductivity κ , which is an analog of the electric conductivity but due to a temperature gradient. It is given by

$$\kappa = \sqrt{\frac{3}{8}} \frac{e}{m\pi} n_e k_B \tau, \quad (25)$$

which is only slightly smaller than the value obtained rigorously, which is $\kappa = en_e k_B \tau / 4m$.

Now we are able to compute the effects due to the sum of the contributions of electric plus thermal gradients

$$\vec{J}_x = \vec{J}_E + \vec{J}_T = \sigma \left(\mathcal{E} + \frac{k_B}{2e} \frac{\partial T}{\partial x} \right). \quad (26)$$

However, along a metal rod one can only expect a transient current, which vanishes when steady state is reached, as positive electric charge accumulates at the hot end and negative charge at the cool end. The equilibrium electric field corresponds to

$$\mathcal{E}_{eq} = -\frac{k_B}{2e} \frac{\partial T}{\partial x}. \quad (27)$$

5. A Quantum Adaptation of Drude's Theory

A realistic description of electron behavior requires the use of Quantum Mechanics, mainly due to the Pauli Exclusion Principle, and the consequent Fermi sea. Electrons, even at room temperature, are constrained in quasi-momentum space to the interior of the Fermi surface. Each quasi-momentum $\vec{p} = \hbar \vec{k}$ is associated to a Bloch function $\psi(\vec{k}, \vec{r})$ of eigen-energy $E(\vec{k})$, and satisfies the Bloch theorem

$$\psi(\vec{k}, \vec{r} + \vec{L}) = \exp(i\vec{k} \cdot \vec{L}) \psi(\vec{k}, \vec{r}), \quad (28)$$

where $\vec{L}$ is a vector that connects two points of the crystal lattice. equation 28 implies that the electron wave function extends over the whole crystal, and therefore it is macroscopically coherent. Accordingly, a weak external electric field only modifies the wave vector $\vec{k} \rightarrow \vec{k}(t)$ without contributing to the electrical resistivity. The electron dynamics is thus governed by

$$\hbar \frac{d\vec{k}}{dt} = -e\vec{\mathcal{E}}. \quad (29)$$

When dissipation is incorporated each k -point is shifted to $\vec{k} \rightarrow \vec{k} + \delta\vec{k}$, where

$$\delta\vec{k} = -e \frac{\vec{\mathcal{E}}\tau}{\hbar}, \quad (30)$$

as illustrated in Fig. 2. Band structure effects can be introduced by replacing m by an effective electron mass m^* ; this leads to a drift velocity given by

$$\vec{v}_{drift} = -e \frac{\vec{\mathcal{E}}\tau}{m^*}, \quad (31)$$

which is compatible with the Drude expression of equation 4. It is worth mentioning that this way other phenomena, like the Pauli susceptibility and the extremal orbits of the de Haas-Van Alphen effect, can also be understood.

By incorporating quantum mechanics and the fermionic electron properties Sommerfeld was able to

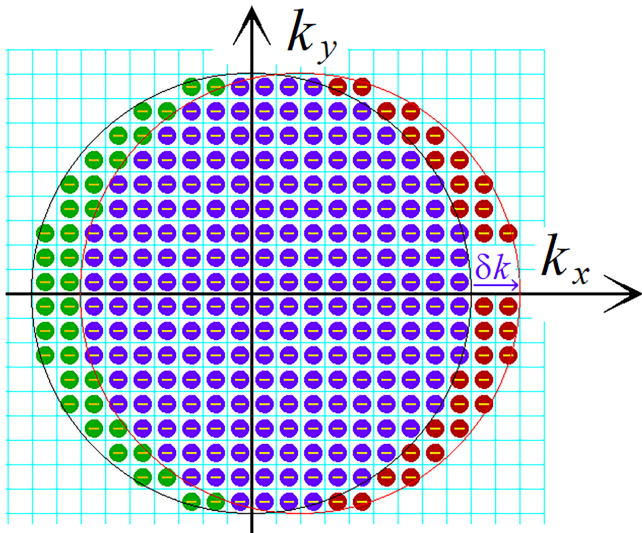


Figure 2: A constant electric field displaces the Fermi surface by $\delta \vec{k} = -e\vec{\mathcal{E}}\tau/\hbar$, with slight overall changes around the Fermi surface; the states marked in green are emptied, and the unoccupied states in red become occupied, generating a current $\propto \vec{\mathcal{E}}$ that obeys Ohm's law.

remove other shortcomings of the Drude theory. For example, incorporating the fact that only the small fraction T/T_F of electrons in the vicinity of the Fermi surface are available to participate in electric and thermal metal dynamics, he gave a final justification to the Drude hypothesis, which was quite “wild” in a 1900 context. Therefore, the Pauli principle limits the effective number of states available for electric and thermal conductivity to N_{eff} given by

$$N_{eff} = N_e \frac{T}{T_F}, \quad (32)$$

also implying that the thermal velocity $v(T)$ has to be replaced by the Fermi velocity v_F , and the mean free path becomes $\lambda = \tau v_F$. The latter implies a correction factor T/T_F for the specific heat of equation 16, and a $(T/T_F)^2$ correction of the electron-electron scattering cross section [3, 20, 21]. Moreover, it justifies the non-interacting electron gas hypothesis adopted by Drude. In 1933 Grüneisen handled explicitly thermal effects, due to the electron-phonon interaction [22], and obtained a quite accurate fit to the $\rho \propto T^5$ for the low temperature, and the $\rho \propto T$ dependence of the of many pure crystalline metals.

6. Summary and Conclusions

Around 1900 physics went through a major revolution, which took a couple of decades to fully appreciate. The analysis of Planck of the electromagnetic radiation spectra, Einstein's theory of Relativity and his explanation of the photoelectric effect in terms of h , the Planck constant, required decades to be recognized as major

milestones. It seems likely that the insights that led Drude to develop his model for metal conductivity were concealed by these achievements. However, as we stress above, it is most remarkable that the assumptions Drude made, in particular the use of the electron relaxation time and the assumption of the existence of an electron gas, turned out to be so accurate. It is also important to stress the importance of Drude's visionary concept of a nearly free electronic gas. Beyond being a useful tool in all Solid State Physics courses, a superposition of a few non-interacting quantum states can account for band structure details of many simple metals. All in all, the contribution by Drude a quarter of a century before the birth of Quantum Mechanics, led to a description of electric conductivity that is still used today, more than 120 years after he conceived it.

Acknowledgments

MK gratefully acknowledges support by ANID FONDECYT 1211902, and CEDENNA through Financiamiento Basal para Centros Científicos y Tecnológicos de Excelencia (grant AFB220001).

References

- [1] P. Drude, *Annalen der Physik* **306**, 566 (1900a).
- [2] H.A. Lorentz, Huygens Institute - Royal Netherlands Academy of Arts and Sciences (KNAW), *Proceedings* **7**, 438 (1905).
- [3] N. Ashcroft and N.D. Mermin, *Solid State Physics* (Saunders College Publishing, New York, 1976).
- [4] H.A. Lorentz, *Elektronentheorie* (Leipzig und Berlin, Teubner, 1909).
- [5] J. Perrin, *Ann. Chemie. Phys.* **18**, 5 (1909).
- [6] R. Newburgh, J. Peidle and W. Rueckner, *Am. J. Phys.* **74**, 478 (2006).
- [7] J.J. Thomson, *Phil. Mag.* **7**, 237 (1904).
- [8] E.K. Lenz, *Annalen der Physik* **34**, 418 (1838).
- [9] E.K. Lenz, *Annalen der Physik* **45**, 105 (1838).
- [10] J.M. Benoit, *Comptes Rendus Hebdo. des Séances de l'Académie des Sci.* **76**, 342 (1873).
- [11] J. Dewar and J.A. Fleming, *Phil. Mag.* **34**, 326 (1892).
- [12] J. Dewar and J.A. Fleming, *Phil. Mag.* **36**, 271 (1893).
- [13] W. Siemens, *Annalen der Physik* **110**, 1 (1860).
- [14] W. Siemens, *Proc. Royal Soc.* **19**, 443 (1870-1871).
- [15] S. Reif-Acherman, *Proceedings of the IEEE* **103**, 713 (2015).
- [16] S. Wroblewski, *Acad. Sci. Paris* **94**, 212 (1882a).
- [17] S. Wroblewski, *Acad. Sci. Paris* **94**, 954 (1882b).
- [18] S. Wroblewski, *Acad. Sci. Paris* **94**, 1355 (1882c).
- [19] P. Drude, *Annalen der Physik* **306**, 1102 (1900b).
- [20] C. Kittel, *Introduction to Solid State Physics* (John Wiley & Sons, New York, 1986), 6 ed.
- [21] C. Kittel, *Quantum theory of solids* (Wiley, New York, 1963).
- [22] E. Grüneisen, *Ann. der Physik* **408**, 530 (1933).